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RENORMALIZED DISSIPATIVE SOLUTIONS OF SECOND ORDER DEGENERATE PARABOLIC BALANCE LAWS

Satoru Takagi *

Abstract

We introduce a new notion of renormalized dissipative solutions for the Cauchy problem of a second order degenerate parabolic balance law $u_t + \operatorname{div}\mathbf{F}(u) = \triangle b(u)$ with locally Lipschitz-continuous flux $\mathbf{F}$ and L^1 data, and prove the equivalence of such solutions and renormalized entropy solutions in the sense of Bendahmane and Karlsen. The structure of renormalized dissipative solutions is flexible and suitable to deal with relaxation systems than the renormalized entropy scheme. The proof of our main theorem is based on the method of doubling variables established by Kružkov.

1 Introduction

We consider the following Cauchy problem:

$$\text{(CP)} \qquad \begin{cases} u_t + \operatorname{div}\mathbf{F}(u) = \triangle b(u) & \text{in} \quad Q := (0,T) \times \mathbf{R}^N, \\ u(0,\cdot) = u_0 & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $T > 0$, $N \geq 1$, $u_0 \in L^1(\mathbf{R}^N)$ is a given function, $\mathbf{F} : \mathbf{R} \to \mathbf{R}^N$ is a locally Lipschitz-continuous flux and $b : \mathbf{R} \to \mathbf{R}$ is a function defined by

$$b(r) := \int_0^r \sigma(s)\, ds \tag{1.1}$$

for given $\sigma \in L^\infty_{loc}(\mathbf{R})^+$ and any $r \in \mathbf{R}$. Here, $L^\infty_{loc}(\mathbf{R})^+$ denotes the space of all nonnegative functions which belong to $L^\infty_{loc}(\mathbf{R})$.

*E-mail: satoru@toki.waseda.jp. The author was supported by Waseda University Grant for Special Research Projects ♯2004A–108.

If the function b is constant, then the diffusion term which is on the right hand side of the equation degenerates, and therefore the equation becomes a hyperbolic scalar conservation law $u_t + \operatorname{div} \mathbf{F}(u) = 0$. It is known that (CP) has many solutions in the sense of distributions called weak solutions. Finding a suitable criterion which would ensure the uniqueness of a weak solution is one of the most interesting problems, and therefore many researchers have studied hyperbolic equations and degenerate parabolic equations including conservation laws. In consequence, various important results have been clarified for the last few decades.

In 1970, Kružkov [16] proved that if $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, then the equation has a unique weak solution $u \in C([0,T); L^1(\mathbf{R}^N)) \cap L^\infty(Q)$ satisfying the entropy inequality, which is the so-called entropy solution. He also introduced the method of doubling variables which is a practical tool and on the basis of the proof of uniqueness. Around three decades later, Chen and Perthame [6] extended the notion of entropy solutions to general degenerate parabolic equations with anisotropic nonlinearity, and obtained uniqueness of an entropy solution by utilizing a kinetic formulation and regularization by convolution. At the same time, Portilheiro [18] defined a dissipative solution of scalar conservation laws with globally Lipschitz-continuous flux $\mathbf{F}$, which was established first by Evans, and showed the equivalence of such solutions and entropy solutions by accretive operator theory. Furthermore, the notion of dissipative solutions was extended by Perthame and Souganidis [17] to the second order degenerate parabolic balance laws and the equivalence result was obtained. The definition of dissipative solutions is more simple and flexible, and also suitable to study asymptotic problems handling relaxation systems than entropy solutions. Direct proofs of existence and uniqueness of dissipative solutions, however, have not been obtained yet.

On the other hand, it is known that if $u_0 \in L^1(\mathbf{R}^N)$, then the mild solution u of (CP) constructed by nonlinear semigroup theory is a unique entropy solution, which is unbounded in general. In the case where $\mathbf{F}$ is only locally Lipschitz-continuous, the flux function $\mathbf{F}(u)$ may fail to be locally integrable since no growth condition is assumed on the flux $\mathbf{F}$, and hence (CP) does not possess a solution even in the sense of distributions. To overcome this, the notion of renormalized entropy solutions has been introduced by Benilan et al. [3] for scalar conservation laws and by Bendahmane and Karlsen [2] for second order degenerate parabolic equations. Furthermore, the existence and uniqueness of a renormalized entropy solution of these equations have been established and the semigroup solutions of (CP) in L^1 spaces are characterized. The arguments in [17] and [18], however, do not work well in the case where $\mathbf{F}$ is only locally Lipschitz-continuous and the solution

u is unbounded. The notion of renormalized solutions has been introduced first by DiPerna and Lions [7] for dealing with the existence of a solution of the Boltzmann equation and utilized for degenerate elliptic and degenerate parabolic problems in the L^1-setting in the last decade. As to the renormalized solutions for degenerate quasilinear elliptic equations including p-Laplace type $g(u) + \operatorname{div} \mathbf{F}(u) = \triangle_p b(u) + f$, see [12] and [15]. Moreover, there are several interesting results for initial-boundary value problems of scalar conservation laws and degenerate elliptic-parabolic-hyperbolic equations $g(u)_t + \operatorname{div} \mathbf{F}(u) = \triangle b(u) + f$. See [4], [5] and [13], for example.

A new concept of renormalized dissipative solutions for a hyperbolic equation with L^1 data has been established in [14] and the equivalence of such solutions and renormalized entropy solutions in the sense of [3] was proved. Existence of renormalized dissipative solutions for a contractive relaxation system describing discrete velocity kinetic models has been also shown in general L^1-settings in [14] and solutions of the system were characterized. The purpose of this paper is to extend this notion to the second order degenerate parabolic balance laws including hyperbolic conservation laws. In Section 2, we recall some important definitions and extend the notion of renormalized dissipative solutions which is a generalization of dissipative solutions in [17]. We next show the equivalence of renormalized dissipative solutions and renormalized entropy solutions in the sense of [2] in Section 3. The anisotropic diffusion case and its applications for certain relaxation systems shall be mentioned in the forthcoming paper [21]. More precisely, we shall consider the Cauchy problem $u_t + \operatorname{div} \mathbf{F}(u) = \operatorname{div}(A(u)\nabla u) + f$, $u(0,\cdot) = u_0$ with L^1 data and deal with the following relaxation systems:

$$
\text{(RS1)} \qquad \begin{cases}
w_t^\varepsilon + \displaystyle\sum_{i=1}^{N} \omega_i V_{n,i} w_{x_i}^\varepsilon = \frac{1}{\varepsilon}\sum_{i=1}^{N}(h_{n,i}(w^\varepsilon) - z_i^\varepsilon) & \text{in } Q, \\
(z_i^\varepsilon)_t - V_{n,i}\,(z_i^\varepsilon)_{x_i} = \dfrac{1}{\varepsilon}(h_{n,i}(w^\varepsilon) - z_i^\varepsilon) & \text{in } Q, \quad i = 1, \cdots, N, \\
w^\varepsilon(0,\cdot) = w_0 & \text{in } \mathbf{R}^N, \\
z_i^\varepsilon(0,\cdot) = z_{i0} & \text{in } \mathbf{R}^N, \quad i = 1, \cdots, N,
\end{cases}
$$

and

$$
\text{(RS2)} \qquad \begin{cases} w_t^\varepsilon + \operatorname{div} \mathbf{F}(w^\varepsilon) - \sum_{i,j=1}^N A_{ij}(w^\varepsilon)_{x_i x_j} &= -\frac{1}{\varepsilon} w^\varepsilon z^\varepsilon \quad \text{in } Q, \\ z_t^\varepsilon &= -\frac{1}{\varepsilon} w^\varepsilon z^\varepsilon \quad \text{in } Q, \\ w^\varepsilon(0,\cdot) &= w_0 \quad \text{in } \mathbf{R}^N, \\ z^\varepsilon(0,\cdot) &= z_0 \quad \text{in } \mathbf{R}^N, \end{cases}
$$

for w^ε and $\mathbf{z}^\varepsilon = (z_1^\varepsilon, \cdots, z_N^\varepsilon)$ with relaxation parameter $\varepsilon > 0$. The first system is interpreted as discrete velocity kinetic models and chemical reaction models. See, for example, [10], [11], [14] and [19]. The second system describes the evolution of a chemical or a biological species called a tracer in a porous medium. This tracer is supposed to be stuck on the surface of the solid frame. See [1] for more details. In [21], we shall also construct a renormalized dissipative solution for the following generalized Stefan problem

$$
\text{(GSP)} \qquad \begin{cases} u_t + \operatorname{div} \mathbf{F}(u^+) - \sum_{i,j=1}^N A_{ij}(u^+)_{x_i x_j} &= 0 \quad \text{in } Q, \\ u(0,\cdot) &= u_0 \quad \text{in } \mathbf{R}^N. \end{cases}
$$

2 The main result

We begin with some notations and definitions. Let $r \in \mathbf{R}$ and $j \in [-1,1]$. We set $r^+ := \max\{r, 0\}$ and $r^- := -\min\{r, 0\}$. Note that $r^- \geq 0$ and $r = r^+ - r^-$. Define a sign function S_j by $S_j(r) = 1$ if $r > 0$, $S_j(r) = -1$ if $r < 0$ or $S_j(0) = j$, and set $S_j^+(r) := \max\{S_j(r), 0\}$.

Let $T_\ell : \mathbf{R} \to [-\ell, \ell]$ denote the truncation function with height $\ell > 0$, that is, $T_\ell(r) := \min\{\max\{r, -\ell\}, \ell\}$ for any $r \in \mathbf{R}$, and we define for $r \in \mathbf{R}$

$$
\beta(r) := \int_0^r \sqrt{\sigma(s)}\, ds,
$$

where $\sigma \in L_{loc}^\infty(\mathbf{R})^+$ satisfies (1.1).

Following [2] we define an entropy-entropy flux triple and renormalized entropy solutions of (CP).

Definition 2.1. *For any convex C^2 entropy function $\eta : \mathbf{R} \to \mathbf{R}$, the corresponding entropy fluxes*

$$\mathbf{q} = (q_1, \cdots, q_N) : \mathbf{R} \to \mathbf{R}^N \quad \text{and} \quad R = (r_{ij}) : \mathbf{R} \to \mathbf{R}^{N\times N}$$

are defined by $q_i'(s) = \eta'(s)\, F_i'(s)$ and $r_{ij}'(s) = \eta'(s)\,\sigma(s)\,\delta_{ij}$ for $i, j = 1, \cdots, N$ and $s \in \mathbf{R}$, where δ_{ij} takes 1 if $i = j$ or 0 otherwise. Then, we define $(\eta, \mathbf{q}, R)$ as an entropy-entropy flux triple.

Definition 2.2. *We say $u \in L^\infty(0, T; L^1(\mathbf{R}^N))$ is a renormalized entropy solution of* (CP) *provided a measurable function $u : Q \to \mathbf{R}^N$ satisfies the following conditions:*

(E1) $\nabla\beta(T_\ell(u)) \in L^2(Q)^N$ for all $\ell > 0$.

(E2) *For any $\ell > 0$ and any entropy-entropy flux triple $(\eta, \mathbf{q}, R)$ with $|\eta'| \leq K$ for some given $K > 0$, there exists a nonnegative bounded Radon measure μ_ℓ^K on Q with $\mu_\ell^K(Q) \to 0$ as $\ell \to \infty$ such that*

$$\begin{aligned} &\eta(T_\ell(u))_t + \operatorname{div} \mathbf{q}(T_\ell(u)) - \sum_{i,j=1}^{N} r_{ij}(T_\ell(u))_{x_i x_j} \\ &\leq \; -\eta''(T_\ell(u)) \left|\nabla\beta(T_\ell(u))\right|^2 + \mu_\ell^K \qquad \text{in } \mathcal{D}'(Q). \end{aligned} \tag{2.1}$$

(E3) *$u(t, \cdot) \to u_0$ in $L^1(\mathbf{R}^N)$ as $t \downarrow 0$ essentially.*

Note that all terms in (2.1) are well-defined since $T_\ell(u) \in L^\infty(Q)$, and also note that (E2) implies there exists a nonnegative bounded Radon measure μ_ℓ on Q with $\mu_\ell(Q) \to 0$ as $\ell \to \infty$ such that $\mu_\ell^K = K_0 \mu_\ell$ with $K_0 := \sup_{r\in[-\ell,\ell]} |\eta'(r)|$. Indeed, for each i, j, putting $\widetilde{\eta} := K_0^{-1}\eta$, $\widetilde{q}_i := K_0^{-1} q_i$ and $\widetilde{r}_{ij} := K_0^{-1} r_{ij}$, this triple $(\widetilde{\eta}, \widetilde{\mathbf{q}}, \widetilde{R})$ should be an entropy-entropy flux triple with $|\widetilde{\eta}'| \leq 1$.

Next, we introduce a new notion of renormalized dissipative solutions which is a generalization of dissipative solutions in the sense of [17].

Definition 2.3. *We say $u \in L^\infty(0, T; L^1(\mathbf{R}^N))$ is a renormalized dissipative solution of* (CP) *if a measurable function $u : Q \to \mathbf{R}^N$ satisfies the following conditions:*

(D1) $\nabla\beta(T_\ell(u)) \in L^2(Q)^N$ for all $\ell > 0$.

(D2) *For any $\ell > 0$, $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with* $\operatorname{spt}\theta \subset (-\ell, \ell)$, *there exists a nonnegative bounded Radon measure ν_ℓ on Q with $\nu_\ell(Q) \to 0$ as $\ell \to \infty$ such that*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\,(T_\ell(u) - k - \xi)^+\,dkdx \\
&\quad\le \int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\,S_0^+(T_\ell(u) - k - \xi)\,\big(-\operatorname{div}\mathbf{F}(k+\xi) + \Delta b(k+\xi)\big)\,dkdx \\
&\qquad - \int_{\mathbf{R}^N} \theta(T_\ell(u) - \xi)\,\Big|\nabla\beta(T_\ell(u)) - \sqrt{\sigma(T_\ell(u))}\,\nabla\xi\Big|^2\,dx \\
&\qquad + \int_{\mathbf{R}^N}\int_{\mathbf{R}} \theta(k)\,S_0^+(\ell - k - \xi)\,dkd\nu_\ell \qquad \text{in } \mathcal{D}'(0,T).
\end{aligned} \tag{2.2}$$

(D3) $u(t,\cdot) \to u_0$ *in* $L^1(\mathbf{R}^N)$ *as* $t \downarrow 0$ *essentially.*

Then we obtain the following main result.

Theorem 2.4. *Suppose that $u \in L^\infty(0,T; L^1(\mathbf{R}^N))$. Then, u is a renormalized entropy solution of* (CP) *if and only if u is a renormalized dissipative solution of* (CP).

Note that if a renormalized entropy (respectively renormalized dissipative) solution u belongs to $L^\infty(Q)$, then it is also an entropy (respectively a dissipative) solution in the sense of [2, Definition 2.2] (respectively [17, Definition 1.3]). As we mentioned in Section 1, uniqueness of an entropy solution in the sense of [2, Definition 2.2] was proved in [6] using a kinetic formulation, and the equivalence of such solutions and dissipative solutions was obtained in [17].

If b is a constant function, then the equation becomes a hyperbolic conservation law $u_t + \operatorname{div}\mathbf{F}(u) = 0$. In this case, the equivalence of renormalized entropy solutions and renormalized dissipative solutions was proved in [14]. Due to appearance of the Dirac mass, however, the definition of renormalized dissipative solutions differs from Definition 2.3. Then we shall reconsider in [21] the contractive relaxation system studied in [14] as an application for the hyperbolic case.

3 Proof of Theorem 2.4

Claim 1: *If $u \in L^\infty(0,T; L^1(\mathbf{R}^N))$ is a renormalized entropy solution of* (CP), *then u is a renormalized dissipative solution.*

Proof. We see from the definition of renormalized entropy solutions that for any $\ell > 0$ and any entropy-entropy flux triple $(\eta, \mathbf{q}, R)$ with $|\eta'| \le K$ for some given $K > 0$, there exists a nonnegative bounded Radon measure μ_ℓ on Q with $\mu_\ell(Q) \to 0$ as $\ell \to \infty$ such that

$$\begin{aligned} 0 \ \le\ & \iint_Q \eta(T_\ell(u))\, \zeta_t \, dxdt + \iint_Q \sum_{i=1}^N q_i(T_\ell(u))\, \zeta_{x_i}\, dxdt + \iint_Q \sum_{i,j=1}^N r_{ij}(T_\ell(u))\, \zeta_{x_i x_j}\, dxdt \\ & - \iint_Q \eta''(T_\ell(u)) \left|\nabla\beta(T_\ell(u))\right|^2 \zeta \, dxdt + \iint_Q K_0 \zeta \, d\mu_\ell \end{aligned} \tag{3.1}$$

for any $\zeta \in C_0^2(Q)^+$, where $K_0 := \sup_{r\in[-\ell,\ell]} \left|\eta'(r)\right|$.

On the other hand, for given $\xi \in C_0^2(\mathbf{R}^N)$ and $\theta \in C_0^2(\mathbf{R})^+$ with $\operatorname{spt}\theta \subset (-\ell, \ell)$, we observe that

$$\eta(T_\ell(u)) \ =\ \int_{\mathbf{R}} (T_\ell(u) - k - \xi(y))^+ \, \theta(k)\, dk$$

is a smooth entropy. Moreover, we see that $\eta'(T_\ell(u)) = \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi(y))\, \theta(k)\, dk$ and $K_0 = \int_{\mathbf{R}} S_0^+(\ell - k - \xi(y))\, \theta(k)\, dk$.

Let ϕ and ρ be standard mollifiers on $\mathbf{R}$ and $\mathbf{R}^N$, respectively. Define ρ_ε by

$$\rho_\varepsilon(x-y) \ :=\ \frac{1}{\varepsilon^N}\, \rho((x-y)/\varepsilon),$$

and let ψ_n be a nonnegative smooth function satisfying

$$\psi_n(t,x) \ :=\ \begin{cases} 1 & \text{if} \ \ |x| \le n \\ 0 & \text{if} \ \ |x| \ge 2n, \end{cases}$$

and $|\nabla\psi_n| \le C/n$ for some $C > 0$. We now recall the definition of an entropy-entropy flux triple and properties of the Dirac mass. Putting $\zeta = \rho_\varepsilon(x-y)\, \phi(t)\, \psi_n(t,x)$ in (3.1), integrating with respect to y over $\mathbf{R}^N$ and using $(\rho_\varepsilon)_{y_i} = -(\rho_\varepsilon)_{x_i}$ yield

$$\begin{aligned} 0 \ \le\ & \int_{\mathbf{R}^N} \iint_Q \eta(T_\ell(u)) \left(\rho_\varepsilon(x-y)\, \phi(t)\, \psi_n(t,x)\right)_t dxdtdy \\ & + \int_{\mathbf{R}^N} \iint_Q \sum_{i=1}^N q_i(T_\ell(u)) \left(\rho_\varepsilon(x-y)\, \phi(t)\, \psi_n(t,x)\right)_{x_i} dxdtdy \\ & + \int_{\mathbf{R}^N} \iint_Q \sum_{i,j=1}^N r_{ij}(T_\ell(u)) \left(\rho_\varepsilon(x-y)\, \phi(t)\, \psi_n(t,x)\right)_{x_i x_j} dxdtdy \\ & - \int_{\mathbf{R}^N} \iint_Q \eta''(T_\ell(u)) \left|\nabla\beta(T_\ell(u))\right|^2 \rho_\varepsilon(x-y)\, \phi(t)\, \psi_n(t,x)\, dxdtdy \\ & + \int_{\mathbf{R}^N} \iint_Q K_0 \rho_\varepsilon(x-y)\, \phi(t)\, \psi_n(t,x)\, d\mu_\ell dy \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,\rho_\varepsilon\Big((T_\ell(u)-k-\xi(y))\,\phi\,\psi_n\Big)_t\,dkdxdtdy \\
&\quad - \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,T_\ell(u)_t\,\theta(k)\,\rho_\varepsilon\,\phi\,\psi_n\,dkdxdtdy \\
&\quad - \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,\mathrm{div}_y\mathbf{F}(k+\xi(y))\,\rho_\varepsilon\,\phi\,\psi_n\,dkdxdtdy \\
&\quad + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k) \\
&\qquad\qquad \times \big(\mathbf{F}(T_\ell(u))-\mathbf{F}(k+\xi(y))\big)\cdot\nabla_x\psi_n\,\rho_\varepsilon\,\phi\,dkdxdtdy \\
&\quad + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi(y))\,\theta(k)\,\triangle_x b(T_\ell(u))\,\rho_\varepsilon\phi\,\psi_n\,dkdxdtdy \\
&\quad + \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi(y))\,\theta(k)\,\rho_\varepsilon\,\phi\,\psi_n\,dkd\mu_\ell dy \\
&=: \sum_{h=1}^{6} I_h^{\varepsilon,n}. \qquad (3.2)
\end{aligned}$$

We begin with $I_5^{\varepsilon,n}$. For $p>0$, we set

$$\omega(p) := \sup\big\{|\xi(x)-\xi(y)|\,;\, x,y\in\mathbf{R}^N,\ |x-y|\le p\big\}.$$

Note that $\omega(p)\ge 0$ for any $p>0$ and $\omega(p)\to 0$ as $p\downarrow 0$. Then we see that

$$\begin{aligned}
I_5^{\varepsilon,n} &\le \iiiint_{\triangle_x b(T_\ell(u))\ge 0} S_0^+(T_\ell(u)-k-\xi(x)+\omega(\varepsilon))\,\triangle_x b(T_\ell(u))\,\theta\,\rho_\varepsilon\phi\,\psi_n\,dkdxdtdy \\
&\quad + \iiiint_{\triangle_x b(T_\ell(u))<0} S_0^+(T_\ell(u)-k-\xi(x)-\omega(\varepsilon))\,\triangle_x b(T_\ell(u))\,\theta\,\rho_\varepsilon\phi\,\psi_n\,dkdxdtdy
\end{aligned}$$

which implies

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} I_5^{\varepsilon,n} &\le \iiint_{\triangle b(T_\ell(u))\ge 0} S_1^+(T_\ell(u)-k-\xi)\,\triangle b(T_\ell(u))\,\theta\,\phi\,\psi_n\,dkdxdt \\
&\quad + \iiint_{\triangle b(T_\ell(u))<0} S_0^+(T_\ell(u)-k-\xi)\,\triangle b(T_\ell(u))\,\theta\,\phi\,\psi_n\,dkdxdt \\
&= \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\triangle b(T_\ell(u))\,\theta\,\phi\,\psi_n\,dkdxdt \\
&\quad + \iiint_{T_\ell(u)=k+\xi} \big(\triangle b(T_\ell(u))\big)^+\,\theta\,\phi\,\psi_n\,dkdxdt.
\end{aligned}$$

As to other integrals, we see from the same arguments as above that

$$\begin{aligned}
I_1^{\varepsilon,n} &= 0,\\
\limsup_{\varepsilon\downarrow 0} I_2^{\varepsilon,n} &\le -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,T_\ell(u)_t\,\theta\,\phi\,\psi_n\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_3^{\varepsilon,n} &\le -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\operatorname{div}\mathbf{F}(k+\xi)\,\theta\,\phi\,\psi_n\,dkdxdt\\
&\quad+\iiint_{T_\ell(u)=k+\xi}\left(\operatorname{div}\mathbf{F}(k+\xi)\right)^-\theta\,\phi\,\psi_n\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_4^{\varepsilon,n} &\le \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\left(\mathbf{F}(T_\ell(u))-\mathbf{F}(k+\xi)\right)\cdot\nabla\psi_n\,\theta\,\phi\,dkdxdt,\\
\limsup_{\varepsilon\downarrow 0} I_6^{\varepsilon,n} &\le \iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi)\,\theta\,\phi\,\psi_n\,dkd\mu_\ell.
\end{aligned}$$

Hence, passing to the limit in (3.2) as $\varepsilon\downarrow 0$ first and then $n\to\infty$ gives

$$\begin{aligned}
0 \le\; & -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,T_\ell(u)_t\,\theta\,\phi\,dkdxdt\\
& -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\operatorname{div}\mathbf{F}(k+\xi)\,\theta\,\phi\,dkdxdt\\
& +\iiint_{T_\ell(u)=k+\xi}\left(\operatorname{div}\mathbf{F}(k+\xi)\right)^-\theta\,\phi\,dkdxdt\\
& +\limsup_{n\to\infty}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\left(\mathbf{F}(T_\ell(u))-\mathbf{F}(k+\xi)\right)\cdot\nabla\psi_n\,\theta\,\phi\,dkdxdt\\
& +\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi)\,\Delta b(T_\ell(u))\,\theta\,\phi\,dkdxdt\\
& +\iiint_{T_\ell(u)=k+\xi}\left(\Delta b(T_\ell(u))\right)^+\theta\,\phi\,dkdxdt+\iint_Q\int_{\mathbf{R}} S_0^+(\ell-k-\xi)\,\theta\,\phi\,dkd\mu_\ell. \qquad (3.3)
\end{aligned}$$

Note that the set $\{k\in(-\ell,\ell)\,;\,\mathcal{L}^{N+1}(\{T_\ell(u)=k+\xi\})=0\}$ is dense in $(-\ell,\ell)$ because $\sum_{k\in C}|k|\,\mathcal{L}^{N+1}(\{T_\ell(u)=k+\xi\})$ is finite for any countable set $C\subset(-\ell,\ell)$, where $\mathcal{L}^{N+1}$ denotes the $(N+1)$-dimensional Lebesgue measure. Hence the cardinality of the set $\{k\in(-\ell,\ell)\,;\,\mathcal{L}^{N+1}(\{T_\ell(u)=k+\xi\})\}$ is at most countable.

We now fix any $k\in(-\ell,\ell)$ and choose a sequence $\{k_n^+\}$ such that $k_n^+\downarrow k$ as $n\to\infty$ and

$\mathcal{L}^{N+1}(\{T_\ell(u) = k_n^+ + \xi\}) = 0$ for any $n \geq 1$. It follows from (3.3) with $k = k_n^+$ that

$$\begin{aligned} 0 \;\leq\; & -\iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, T_\ell(u)_t\, \theta\, \phi\, dkdxdt \\ & -\iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \operatorname{div} \mathbf{F}(k+\xi)\, \theta\, \phi\, dkdxdt \\ & + \limsup_{n\to\infty} \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \big(\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi)\big) \cdot \nabla\psi_n\, \theta\, \phi\, dkdxdt \\ & + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \Delta b(T_\ell(u))\, \theta\, \phi\, dkdxdt + \iint_Q \int_{\mathbf{R}} S_0^+(\ell - k - \xi)\, \theta\, \phi\, dkd\mu_\ell \\ =: & \sum_{m=1}^{5} I_m. \end{aligned} \tag{3.4}$$

Using the properties of the Dirac mass, we have

$$I_1 = \iint_Q \int_{\mathbf{R}} \big(T_\ell(u) - k - \xi\big)^+\, \theta\, \phi'\, dkdxdt.$$

As to I_3, we first note that $\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi) + \mathbf{F}(k) \in L^1(Q)^N$. From this, we see that

$$\lim_{n\to\infty} \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi) \big(\mathbf{F}(T_\ell(u)) - \mathbf{F}(k+\xi) + \mathbf{F}(k)\big) \cdot \nabla\psi_n\, \theta\, \phi\, dkdxdt = 0.$$

On the other hand, thanks to Chebyshev's inequality, we have for $k > 0$ that

$$\mathcal{L}^{N+1}\big(\{T_\ell(u) - \xi > k\}\big) \;\leq\; \frac{1}{k} \iint_Q \big|T_\ell(u) - \xi\big|\, dxdt \;<\; \infty,$$

and therefore we see that

$$\lim_{n\to\infty} \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \mathbf{F}(k) \cdot \nabla\psi_n\, \theta\, \phi\, dkdxdt \;=\; 0.$$

For $k < 0$, the same result can be also obtained. From these observations, we conclude that $I_3 = 0$. We now calculate I_4 as

$$\begin{aligned} I_4 \;=\; & \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \Delta\big(b(T_\ell(u)) - b(k+\xi)\big)\, \theta\, \phi\, dkdxdt \\ & + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \Delta b(k+\xi)\, \theta\, \phi\, dkdxdt \\ =\; & -\iint_Q \theta(T_\ell(u) - \xi) \left|\nabla\beta(T_\ell(u)) - \sqrt{\sigma(T_\ell(u))}\, \nabla\xi\right|^2 \phi\, dxdt \\ & + \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\, \Delta b(k+\xi)\, \theta\, \phi\, dkdxdt. \end{aligned}$$

Combining these estimates, we obtain that

$$
\begin{aligned}
0 \;\le\;& \iint_Q \int_{\mathbf{R}} (T_\ell(u) - k - \xi)^+ \,\theta\, \phi' \, dkdxdt + \iint_Q \int_{\mathbf{R}} S_0^+(\ell - k - \xi)\,\theta\,\phi\, dkd\mu_\ell \\
&+ \iint_Q \int_{\mathbf{R}} S_0^+(T_\ell(u) - k - \xi)\,\big(- \operatorname{div}\mathbf{F}(k+\xi) + \Delta b(k+\xi)\big)\,\theta\,\phi\, dkdxdt \\
&- \iint_Q \theta(T_\ell(u) - \xi)\left|\nabla\beta(T_\ell(u)) - \sqrt{\sigma(T_\ell(u))}\,\nabla\xi\right|^2 \phi\, dxdt.
\end{aligned}
$$

This is exactly (D2). □

Claim 2: *If $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ is a renormalized dissipative solution of* (CP)*, then u is a renormalized entropy solution.*

Proof. Let $u \in L^\infty(0,T;L^1(\mathbf{R}^N))$ be a renormalized dissipative solution of (CP). We consider a function $\alpha \in C_0^2(\mathbf{R}^N)^+$ and for each $\varepsilon,\ \lambda > 0$ a nondecreasing smooth function $\xi_{\varepsilon,\lambda}$ defined by

$$
\xi_{\varepsilon,\lambda}(r) := \begin{cases} 0 & \text{for } |r| \le \lambda \\ \text{strictly increasing} & \text{for } \lambda \le |r| \le \lambda + \varepsilon \\ 1/\varepsilon & \text{for } |r| \ge \lambda + \varepsilon. \end{cases}
$$

Let $V(N)$ denote the volume of the unit ball in $\mathbf{R}^N$. Using the test function $\xi_{\varepsilon,\lambda}(x-y)$ in (2.2), multiplying by $\alpha_\lambda(y) := \alpha(y)\, V(N)^{-1}\, \lambda^{-N}$ and integrating with respect to y yield for any $\phi \in C_0^1(0,T)^+$,

$$
\begin{aligned}
0 \;\le\;& \int_{\mathbf{R}^N}\iint_Q \int_{\mathbf{R}} \theta(k)\,(T_\ell(u) - k - \xi_{\varepsilon,\lambda}(x-y))^+ \,\phi'(t)\,\alpha_\lambda(y)\, dkdxdtdy \\
&- \int_{\mathbf{R}^N}\iint_Q \int_{\mathbf{R}} \theta(k)\, S_0^+(T_\ell(u) - k - \xi_{\varepsilon,\lambda}(x-y)) \\
&\qquad\qquad \times \operatorname{div}_x \mathbf{F}(k + \xi_{\varepsilon,\lambda}(x-y))\,\phi(t)\,\alpha_\lambda(y)\, dkdxdtdy \\
&+ \int_{\mathbf{R}^N}\iint_Q \int_{\mathbf{R}} \theta(k)\, S_0^+(T_\ell(u) - k - \xi_{\varepsilon,\lambda}(x-y)) \\
&\qquad\qquad \times \Delta_x b(k + \xi_{\varepsilon,\lambda}(x-y))\,\phi(t)\,\alpha_\lambda(y)\, dkdxdtdy \\
&- \int_{\mathbf{R}^N}\iint_Q \theta(T_\ell(u) - \xi_{\varepsilon,\lambda}(x-y)) \\
&\qquad\qquad \times \left|\nabla_x\beta(T_\ell(u)) - \sqrt{\sigma(T_\ell(u))}\,\nabla_x\xi_{\varepsilon,\lambda}(x-y)\right|^2 \phi(t)\,\alpha_\lambda(y)\, dxdtdy \\
&+ \int_{\mathbf{R}^N}\iint_Q \int_{\mathbf{R}} \theta(k)\, S_0^+(\ell - k - \xi_{\varepsilon,\lambda}(x-y))\,\phi(t)\,\alpha_\lambda(y)\, dkd\nu_\ell dy \\
=:\;& \sum_{m=1}^{5} J_m^{\varepsilon,\lambda}.
\end{aligned}
\tag{3.5}
$$

We begin with $J_1^{\varepsilon,\lambda}$. Thanks to the Lebesgue differentiation theorem, we have

$$\begin{aligned}
J_1^{\varepsilon,\lambda} &= \iiiint_{\xi_{\varepsilon,\lambda}=0} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha_\lambda(y)\,dkdxdtdy \\
&\quad + \iiiint_{\xi_{\varepsilon,\lambda}>0} (T_\ell(u)-k-\xi_{\varepsilon,\lambda})^+\,\theta(k)\,\phi'\,\alpha_\lambda(y)\,dkdxdtdy \\
&\to \iiiint_{|x-y|\le\lambda} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha_\lambda(y)\,dkdxdtdy \qquad (\varepsilon\downarrow 0) \\
&= \iint_Q\int_{\mathbf{R}} \frac{1}{V(N)\,\lambda^N}\int_{|x-y|\le\lambda} \alpha(y)\,dy\,(T_\ell(u)-k)^+\,\theta(k)\,\phi'\,dkdxdt \\
&\to \iint_Q\int_{\mathbf{R}} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha(x)\,dkdxdt \quad (\lambda\downarrow 0).
\end{aligned}$$

Let $\Theta'(\cdot) := \theta(\cdot)$ with $\Theta(-\infty)=0$. Calculating other integrals similarly, we obtain that

$$\begin{aligned}
J_2^{\varepsilon,\lambda} &\to -\iiiint_{|x-y|\le\lambda} S_0^+(T_\ell(u)-k)\,\mathbf{F}(k)\cdot\nabla_y\alpha_\lambda(y)\,\theta(k)\,\phi\,dkdxdtdy \\
&\quad + \iiint_{|x-y|\le\lambda} \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla_y\alpha_\lambda(y)\,\phi\,dxdtdy \quad (\varepsilon\downarrow 0) \\
&\to -\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\,\mathbf{F}(k)\cdot\nabla_x\alpha_\lambda(x)\,\theta(k)\,\phi\,dkdxdt \\
&\quad + \iint_Q \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla_x\alpha_\lambda(x)\,\phi\,dxdt \quad (\lambda\downarrow 0) \\
&= \iint_Q \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla\alpha(x)\,\phi\,dxdt,
\end{aligned}$$

$$\begin{aligned}
&J_3^{\varepsilon,\lambda} + J_4^{\varepsilon,\lambda} \\
&= \int_{\mathbf{R}^N}\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k-\xi_{\varepsilon,\lambda})\big(b(k+\xi_{\varepsilon,\lambda})-b(T_\ell(u))\big)\,\theta(k)\,\phi\,\Delta_y\alpha_\lambda(y)\,dkdxdtdy \\
&\quad - \iint_Q\int_{\mathbf{R}} \theta(T_\ell(u)-\xi_{\varepsilon,\lambda})\,\left|\nabla_x\beta(T_\ell(u))\right|^2\phi\,\alpha_\lambda(y)\,dxdtdy \\
&\quad + 2\iint_Q\int_{\mathbf{R}} \theta(T_\ell(u)-\xi_{\varepsilon,\lambda})\,\sqrt{\sigma(T_\ell(u))}\,\nabla_x\beta(T_\ell(u))\cdot\nabla_x\xi_{\varepsilon,\lambda}\phi\,\alpha_\lambda(y)\,dxdtdy \\
&\to \iiiint_{|x-y|\le\lambda} S_0^+(T_\ell(u)-k)\,\big(b(k)-b(T_\ell(u))\big)\,\theta(k)\,\phi\,\Delta_y\alpha_\lambda(y)\,dkdxdtdy \\
&\quad - \iiint_{|x-y|\le\lambda} \theta(T_\ell(u))\,\left|\nabla_x\beta(T_\ell(u))\right|^2\phi\,\alpha_\lambda(y)\,dxdtdy \\
&\quad - 2\iiint_{|x-y|\le\lambda} \Theta(T_\ell(u))\,\sqrt{\sigma(T_\ell(u))}\,\nabla_x\beta(T_\ell(u))\cdot\nabla_y\alpha_\lambda(y)\phi\,dxdtdy \quad (\varepsilon\downarrow 0)
\end{aligned}$$

$$
\begin{aligned}
\to\ & \iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\left(b(k)-b(T_\ell(u))\right)\theta(k)\,\phi\,\Delta\alpha(x)\,dkdxdt \\
& -\iint_Q \theta(T_\ell(u))\left|\nabla\beta(T_\ell(u))\right|^2\phi\,\alpha(x)\,dxdt \\
& -2\iint_Q \Theta(T_\ell(u))\,\sqrt{\sigma(T_\ell(u))}\,\nabla\beta(T_\ell(u))\cdot\nabla\alpha(x)\phi\,dxdt \quad (\lambda\downarrow 0)
\end{aligned}
$$

and

$$
\begin{aligned}
J_5^{\varepsilon,\lambda}\ \to\ & \iiiint_{|x-y|\le\lambda} S_0^+(\ell-k)\,\theta(k)\,\phi\,\alpha_\lambda(y)\,dkd\nu_\ell dy \qquad (\varepsilon\downarrow 0) \\
\to\ & \iint_Q\int_{\mathbf{R}} S_0^+(\ell-k)\,\theta(k)\,\phi\,\alpha(x)\,dkd\nu_\ell \quad (\lambda\downarrow 0).
\end{aligned}
$$

Combining these estimates, (3.5) can be written as

$$
\begin{aligned}
0\ \le\ & \iint_Q\int_{\mathbf{R}} (T_\ell(u)-k)^+\,\theta(k)\,\phi'\,\alpha(x)\,dkdxdt \\
& +\iint_Q \Theta(T_\ell(u))\,\mathbf{F}(T_\ell(u))\cdot\nabla\alpha(x)\,\phi\,dxdt \\
& +\iint_Q\int_{\mathbf{R}} S_0^+(T_\ell(u)-k)\left(b(k)-b(T_\ell(u))\right)\theta(k)\,\phi\,\Delta\alpha(x)\,dkdxdt \\
& -\iint_Q \theta(T_\ell(u))\left|\nabla\beta(T_\ell(u))\right|^2\phi\,\alpha(x)\,dxdt \\
& -2\iint_Q \Theta(T_\ell(u))\,\sqrt{\sigma(T_\ell(u))}\,\nabla\beta(T_\ell(u))\cdot\nabla\alpha\phi\,dxdt \\
& +\iint_Q\int_{\mathbf{R}} S_0^+(\ell-k)\,\theta(k)\,\phi\,\alpha(x)\,dkd\nu_\ell. \qquad (3.6)
\end{aligned}
$$

Following the definition of an entropy-entropy flux triple, we see that

$$
\begin{aligned}
\eta(T_\ell(u)) &= \int_{\mathbf{R}} \eta''(k)\,(T_\ell(u)-k)^+\,dk, \\
\eta'(T_\ell(u)) &= \int_{\mathbf{R}} \eta''(k)\,S_0^+(T_\ell(u)-k)\,dk, \\
q_i(T_\ell(u))_{x_i} &= \eta'(T_\ell(u))\,F_i(T_\ell(u))_{x_i}, \\
r_{ij}(T_\ell(u))_{x_ix_j} &= \eta'(T_\ell(u))_{x_i}\,b(T_\ell(u))_{x_i}+\eta'(T_\ell(u))\,b(T_\ell(u))_{x_ix_i}.
\end{aligned}
$$

Putting $\theta=\eta''$ and $\Theta=\eta'$ in (3.6), we obtain that

$$
\begin{aligned}
0\ \le\ & \iint_Q \eta(T_\ell(u))\,\phi'\,\alpha\,dxdt+\iint_Q \mathbf{q}(T_\ell(u))\cdot\nabla\alpha\,\phi\,dxdt \\
& +\iint_Q \sum_{i,j=1}^N r_{ij}(T_\ell(u))\,\alpha_{x_ix_j}\,\phi\,dxdt \\
& -\iint_Q \eta''(T_\ell(u))\left|\nabla\beta(T_\ell(u))\right|^2\phi\,\alpha\,dxdt+\iint_Q \eta'(\ell)\,\phi\,\alpha\,d\nu_\ell, \qquad (3.7)
\end{aligned}
$$

which is exactly (E2). Thus we complete the proof of the theorem. □

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References

[1] M. Belhadj, J.-F. Gerbeau and B. Perthame, *A multiscale colloid transport model with anisotropic degenerate diffusion*, Asymptot. Anal. **34** (2003), 41–54.

[2] M. Bendahmane and K. H. Karlsen, *Renormalized entropy solutions for quasilinear anisotropic degenerate parabolic equations*, SIAM J. Math. Anal. **36** (2004), 405–422.

[3] Ph. Benilan, J. Carrillo and P. Wittbold, *Renormalized entropy solutions of scalar conservation laws*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 313–327.

[4] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Rational Mech. Anal. **147** (1999), 269–361.

[5] J. Carrillo and P. Wittbold, *Renormalized entropy solutions of scalar conservation laws with boundary condition*, J. Differential Equations **185** (2002), 137–160.

[6] G.-Q. Chen and B. Perthame, *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations*, Ann. Inst. H. Poincare Anal. Non Lineaire **20** (2003), 645–668.

[7] R. J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: Global existence and weak stability*, Ann. of Math. **130** (1989), 321–366.

[8] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A **111** (1989), 359–375.

[9] L. C. Evans and R. F. Gariepy, "Measure Theory and Fine Properties of Functions", CRC Press, Boca Raton, 1992.

[10] M. A. Katsoulakis and A. E. Tzavaras, *Contractive relaxation systems and the scalar multidimensional conservation law*, Comm. Partial Differential Equations **22** (1997), 195–233.

[11] M. A. Katsoulakis and A. E. Tzavaras, *Multiscale analysis for interacting particles: relaxation systems and scalar conservation laws*, J. Statist. Phys. **96** (1999), 715–763.

[12] K. Kobayasi, *Existence of renormalized solutions of degenerate quasilinear elliptic equations*, Academic Studies, Series of Mathematics, School of Education, Waseda University **47** (1998), 29–46.

[13] K. Kobayasi, *The equivalence of weak solutions and entropy solutions of nonlinear degenerate second-order equations*, J. Differential Equations **189** (2003), 383–395.

[14] K. Kobayasi and S. Takagi, *An equivalent definition of renormalized entropy solutions for scalar conservation laws*, Differential Integral Equations **18** (2005), 19–33.

[15] K. Kobayasi, S. Takagi and T. Uehara, *Uniqueness of renormalized solutions of degenerate quasilinear elliptic equations*, Academic Studies, Series of Mathematics, School of Education, Waseda University **49** (2000), 5–15.

[16] S. N. Kružkov, *First order quasilinear equations with several independent variables*, Math. USSR-Sb. **10** (1970), 217–243.

[17] B. Perthame and P. E. Souganidis, *Dissipative and entropy solutions to non-isotropic degenerate parabolic balance laws*, Arch. Rational Mech. Anal. **170** (2003), 359–370.

[18] M. Portilheiro, *Weak solutions for equations defined by accretive operators I*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), 1193–1207.

[19] M. Portilheiro, *Weak solutions for equations defined by accretive operators II*, J. Differential Equations **195** (2003), 66–81.

[20] S. Takagi, *On renormalized dissipative solutions for parabolic-hyperbolic equations*, to appear in GAKUTO Internat. Ser. Math. Sci. Appl.

[21] S. Takagi, *Renormalized dissipative solutions for quasilinear anisotropic degenerate parabolic equations*, preprint.